"That Which Surpasses All Understanding": The Limitations of Human Thought

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One generation passeth away, and another generation cometh: but the earth abideth for ever. The sun also ariseth, and the sun goeth down, and hasteth to his place where he arose. The wind goeth toward the south, and turneth about unto the north; it whirleth about continually, and the wind returneth again according to his circuits. All the rivers run into the sea; yet the sea is not full; unto the place from whence the rivers come, thither they return again. All things are full of labour; man cannot utter it: the eye is not satisfied with seeing, nor the ear filled with hearing.—Ecclesiastes 1:4–8

I remember those verses striking a powerful chord within me when I read them on a bright autumn day in 1980. I was then in the first few months of my LDS mission in central Virginia. But reading those words took my mind and emotions back to the desert mountains of western Utah earlier that year. A friend and I had taken a quick camping trip to collect fossils in that remote area; and something in the desert sun, the bare exposure of earth, and the surrounding evidence of unimaginably ancient life produced a feeling so strong that I recognized it immediately when I later stumbled on that passage of scripture. I couldn't then put my finger on the exact meaning of the emotion—something about the smallness of our place in the universe and our inability to understand it all. It was as powerful as any religious feeling I had ever had, and its duplication at reading the opening of Ecclesiastes

nearly brought me to tears. I read the remainder of the book eagerly, naively hoping to find its resolution.

That same yearning sense of inadequacy returned to me powerfully again several years later on another bright autumn afternoon. This time I was a graduate student pursuing my doctorate in mathematics at the University of Washington in Seattle. I was at the point in my degree program where only the dissertation research remained, so my afternoons were usually spent sitting at my desk scribbling on scratch paper and looking for some significant idea to break. That afternoon I chose to leave my desk and instead enjoy the sunshine outside. Taking my scratch pad and pencil with me, I walked to a quiet area of campus and settled in to work. My research project in geometry involved a technical question about tilings—the filling up of space by geometric shapes. My scratch pads would fill with patterns of tiles and formulae attempting to explain their properties. The emotion I've been trying to describe hit me that day as my eyes changed focus from the pad in my hands to the leaf-tiled ground underneath me. Despite my supposed sophistication in mathematical reasoning, I was only toying with docile patterns. All around me lay complexity I could never capture in any formula. I picked up a single colored leaf, gazing at the intricate veining on its face, and the feeling deepened. My usual pride in thinking of mathematics as a search for pure and ultimate truth faltered as I realized that the patterns I studied were the faintest shadows of an indescribable reality.

That day the way I looked at mathematics changed, and a real interest in the relationship between my faith and my scholarship began. My research interests have remained in geometry, but I have been fascinated by the philosophy inspired and informed by modern mathematics. I've been particularly impressed that some mathematics can touch in me the same chord that Ecclesiastes strikes. In fact, there is a good deal of interesting mathematics that relates directly to those same limits on human understanding. My musings begin with two cautionary notes:

1. What I say here will involve some speculation, both from a theological and a mathematical viewpoint. However, what I say is consistent with current knowledge in mathematics—that is, it is at least within the realm of possibility as far as we presently know. I believe it is similarly consistent with Latter-day Saint doctrine;

while it may be speculative, it contains nothing contradictory to standard Church teachings.

2. The mathematics necessary for this exposition is surprisingly accessible in its general ideas, but there will be some terminology and concepts with which non-mathematicians will not be familiar. Please be patient, and rest assured that we won't have to deal with any actual equation-chasing or number-crunching. What is described here is more meta-mathematics than mathematics itself, not too difficult for a careful reader to follow.

We will return to the theological implications eventually. But first, I offer a (reasonably) quick introduction to some background concepts we'll need.

Historical Background

To understand *what* mathematics says about the limits of human reasoning, it is nearly essential to understand *how* it is that mathematics even came to address such topics.

Numerical calculations were done by several cultures as early as before 3000 B.C. But it was only when the Greeks introduced the philosophical notion of proof in about 600 B.C. that we had true mathematics. For however one chooses to define mathematics (a notoriously difficult task), the use of deductive reasoning to draw conclusions from a set of assumptions is at its heart. Thales (624–548 B.C.) supposedly wrote the first proofs; and by Euclid's time (about 300 B.C.), the Greeks had evolved the axiomatic method, a formalization of the deductive process in which a small set of assumptions (axioms) is set forth initially and then a superstructure of proven facts (theorems) is built up from that foundation. Most of us know Euclid's name from its association with geometry. But the *Elements* (the work for which he is primarily known) is most notable for its remarkable success in its use of the axiomatic method.

The success of Euclid's *Elements* helped to solidify this method as *the* way to do mathematics. Precision in stating and tracking assumptions became the gold standard by which mathematical works are judged. Modern mathematics has taken the axiomatic method to new heights of formality, but the method remains the same: Begin by stating your axioms, then work carefully within the laws of logic to prove the consequences of those as-

sumptions. So fundamental is the axiomatic method to the discipline of mathematics that, just as the sciences are distinguished by their use of the scientific method, one could characterize mathematics as the use of the axiomatic method.

The growth of mathematical understanding in the 2,300 years since Euclid has included long periods of stagnation, false starts, and even regression. There have also been swings in perceptions of the degree to which mathematics can accurately tell us about reality. The Pythagoreans (a semi-religious cult founded by Pythagoras in the sixth century B.C.) believed in the creed that "all is number." They believed that literally all of observable reality could be explained by the properties of the natural numbers (1, 2, 3, . . .) and their ratios. However, medieval scientists developed new methods of empirical observation more quickly than mathematical principles to explain those observations were advanced. As a result, faith that the universe could be described mathematically gave way to fear that its workings might be undiscoverable to the human mind. And while the tool of algebra that emerged in the thirteenth and fourteenth centuries A.D. was put to impressive use, its applications were limited in scope. The hope of answering big questions through mathematical reasoning would reemerge only after the Enlightenment.

The invention of calculus in the late seventeenth century marks a turning point, not just in mathematics, but also in human intellectual progress. A century before Newton, the prevailing worldview was laced with superstition. Humans observed a universe whose order remained essentially mysterious to them. But the generations following Newton, with *Principia* in hand, saw a clockwork universe operating according to rules that were describable in mathematical terms. Once again, mathematicians worked with the exuberant hope of a complete mathematical description of everything. The train of deduction the Greeks had set in motion was back on track and running with a full head of steam.

But the train had gotten a bit ahead of its own engine. Much of the voluminous work done by the great mathematicians of the eighteenth century was lacking in the rigor usually associated with the axiomatic method. It was as if the mathematics community was impatient with the slow development of rigorous methods

and could not be held back from exploring the exciting new vistas opened by the calculus. The work of justification could be done after the adrenaline rush.

That time came in the latter half of the nineteenth century. But as usually happens, the work of justification brought many difficulties and generated more questions than were actually answered. As early as the middle of the nineteenth century came indications that the tracks on which our mathematical train was traveling were not headed toward a complete description of the physical universe. Mathematicians generated non-Euclidean geometries, complete with bizarre and counterintuitive theorems, but with an internal logic as mathematically consistent as Euclid's revered *Elements*. Mathematicians realized that the axiomatic method could be applied to many different sets of axioms, giving rise to many different mathematical universes, all of which were internally consistent, and none of which could claim to be a perfect model of physical reality. The discipline of mathematics began to chart a more independent course aimed toward abstraction rather than analytical modeling of physical phenomena.

Infinity Rears Its Head

And then there was the problematic concept of infinity. It runs all through calculus, as any freshman calculus student today can tell you. But because the details of calculus's development had been postponed, the exact nature of the "infinitesimal" numbers it used had not been dealt with. It was not until the late nineteenth century (two hundred years after Newton!) that the German mathematician Karl Weierstrass finally provided calculus with a rigorous base and did so without resorting to a new mathematics of infinity. But the suspicion remained that infinity would need to be conquered.

In 1874 Georg Cantor published a paper announcing the beginning of the battle for infinity. The first startling conclusion from this paper was that there are different "sizes" of infinities, an idea that remains as counterintuitive today as it was when Cantor first announced it.[1] The basic idea is this: the "sizes" (cardinalities) of two sets are compared by considering one-to-one correspondences between the sets. If set X can be put into one-to-one

correspondence with a part of set Y, then we say that the cardinality of Y is at least as great as that of X—that is, $|Y| \geq |X|$. (Here I use the symbol $|X|$ to denote the cardinal number of set X.) This device certainly works for finite cardinal numbers and, in fact, is the way we intuitively learn to think of numbers as children: 3 is less than 7 because we can associate a set of three objects in a one-to-one way with only part of a set of seven objects.

Cantor's breakthrough was to apply this same simple principle to infinite sets: If X and Y are sets with infinitely many elements each, we may still compare the sizes of X and Y by asking if X can be put into one-to-one correspondence with part of Y. If this is possible, then we can still write $|Y| \geq |X|$, just as we do in the finite case. If both $|Y| \geq |X|$ and $|X| \geq |Y|$ are true (that is, if X can be put into one-to-one correspondence with part of Y *and* Y can be put into one-to-one correspondence with part of X), then we conclude that $|Y| = |X|$—the sets have equal cardinality.[2] However, if $|Y| \geq |X|$ holds true, but $|X| \geq |Y|$ is *not* true, then we conclude $|Y| > |X|$. The set Y is strictly larger (of greater cardinality) than X. Cantor managed to show that $|\mathbb{R}| > |\mathbb{N}|$ where $\mathbb{R}$ is the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers. The set of real numbers $\mathbb{R}$ includes all numbers most of us ever think about—all those that can be written in decimal form, even if the decimal expansion never ends. The natural numbers are obviously in one-to-one correspondence with part of the real numbers; they *are* part of the real numbers, after all. But Cantor proved that there can be no one-to-one correspondence between the real numbers and any set of natural numbers, so $|\mathbb{R}| > |\mathbb{N}|$. Both sets are infinite, but they are not of equal cardinality.

Mathematicians use the symbol $\aleph_0$ (pronounced "aleph naught"—aleph is the first character in the Hebrew alphabet) to denote $|\mathbb{N}|$, the cardinal number of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. A set whose cardinal number is $\aleph_0$ is said to be countable, since putting a set in exact one-to-one correspondence with $\{1, 2, 3, \ldots\}$ can be thought of as "counting" that set. But according to Cantor's work, there are cardinal numbers larger than $\aleph_0$. In fact, he showed that there are infinitely many infinite cardinal numbers and that there is no largest cardinal number. Any set (such as $\mathbb{R}$) whose cardinal number is larger than $\aleph_0$ is said to be uncountable. Informally,

countable sets are "small" infinite sets—much smaller than un-countable sets.

Types of Numbers

A second major implication of Cantor's paper is a strong theme running through modern mathematics—that most numbers are very strange. To understand this idea, we first need to set out some different classes of numbers. Mathematicians like to classify numbers according to the types of equations for which they might be solutions. Although this system may seem odd at first, it is supported by centuries of experience. And it does make sense, for even informally we tend to think of numbers as solutions to equations. If I were to ask you what $2/3$ is, you might respond that it is the result of the quantity 2 being divided into 3 equal pieces: In other words, it is a quantity x, 3 of which would equal 2—a solution to the algebraic equation $3x = 2$.

Now equations like this one, of the form $ax = b$ where a and b are integers, are called linear equations. Their solutions (the fractions b/a) are called rational numbers. Numbers that are not rational—those that cannot be written as fractions of integers—are called, of course, irrational. Despite the sinister-sounding name, many irrational numbers are actually quite familiar to us. For instance, $\sqrt{2}$ is irrational. But while $\sqrt{2}$ may not be the solution to a linear equation, it is the solution to an only slightly more complex equation, namely $x^2 = 2$. This simple equation gives us a concrete way to think of $\sqrt{2}$; so despite being irrational, it is still fairly understandable.

Generally, a number x is said to be an algebraic number if it is the solution to a polynomial equation $a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x = b$ where n is some positive integer and a_n, a_{n-1}, ..., a_2, a_1, and b are all integers. So $\sqrt{2}$, despite being irrational, is definitely algebraic. Numbers that are not algebraic are called transcendental. You might be more familiar with the rational/irrational split of real numbers than with the algebraic/transcendental split. But in terms of characterizing which numbers are understandable and which are not, the latter does a much better job. (Again, because we understand numbers in terms of equations, transcendental numbers are not solutions to nice equations, so in a very real

sense, we have no fundamental way to grasp them.) You probably can't name any transcendental numbers other than a very few famous examples like the number π. (Some readers may also be familiar with the number e). It isn't that you don't know enough math to know more transcendentals; it's just that most transcendentals are so bizarre in their makeup as to be beyond human description.

Now, back to Cantor. What Cantor's results proved is that the "nice" algebraic numbers, while infinite in cardinality, form a smaller infinity than the "messy" transcendental numbers. In fact, the algebraic numbers are countable while the transcendental numbers are uncountable. This difference is great enough that, if you choose a truly random real number, the probability that it will be algebraic is zero. Oddly (and disturbingly to Cantor's contemporaries), Cantor accomplished this proof without giving any way of actually generating transcendental numbers. In effect, his conclusion means that almost all real numbers are too strange for us to "see."

Twentieth-century mathematics gave us another division of numbers into two classes—a division that is even more fundamental to the question of what it means to "understand" a number. The ideas came from the theory of computation—a mathematical exploration of what computing machines (represented in this case simply as sets of rules for manipulating inputs into outputs) can and cannot do. The groundwork of this theory was laid even before the development of electronic digital computers. There are several abstract models of computing machines, but the most widely accepted model is called a Turing machine[3] (TM). A TM can best be understood as a mathematical model for an algorithmic process. Any computer running any program, and even human decision-making processes, can (in theory) be modeled by a TM.

We say that a number x is computable if there exists a TM that can output x to any decimal accuracy we wish. Clearly any rational number is computable, since we can output the decimal expansion of a/b by the simple algorithmic process of long division. In fact, you probably won't have too much trouble believing that all algebraic numbers are computable (i.e., the polynomial equation

that defines an algebraic number can be turned into a method for generating its decimal expansion). But the class of computable numbers is even bigger than the set of algebraic numbers since many transcendental numbers are also computable. For instance, the most famous transcendental number, π, can be computed using the following striking fact from calculus:

$$\pi = 4 - 4/3 + 4/5 - 4/7 + 4/9 - 4/11 + 4/13 - 4/15 + 4/17 - 4/19 + \ldots$$

This does not mean we could ever write down the entire decimal expansion of π. We certainly cannot do that, for we know that it continues forever with no repetition or apparent pattern. But if you want to know the three-millionth digit[4] after the decimal point in π, it could be computed by this formula.[5] Since *any* method by which we choose to create a decimal number could be modeled by a TM, the computable numbers are the only numbers we can ever hope to "name" or write down. By definition, you can never write down a non-computable number.

But, as you may have already guessed, most real numbers are non-computable. In fact, the set of all TMs turns out to be a countable set, according to an argument we need not go into here. But the set of computable numbers has cardinality no bigger than the set of TMs, since there is an obvious one-to-one correspondence between the computable numbers and a collection of TMs. So, since the real numbers are uncountable and the computable numbers are just a puny countable part of all real numbers, in a very exact way we can say that "almost all" real numbers are uncomputable, and thus beyond our comprehension.

The Law of Mathematical Unapproachability

The second discovery from Cantor's famous paper—that most real numbers are strange—is a precursor to a broad theme in modern mathematics, which I call the Law of Mathematical Unapproachability. It can be simply stated as: "Most objects in the universe of mathematics are too wild for humans to describe." What is the "universe of mathematics"? Most mathematicians inherently believe that there *is* such a thing,[6] though they would be hard pressed to describe it to you.[7] But this universe of mathematics is as much a place to a mathematician as any physical location you've ever visited, even though mathematicians "go there" only

mentally through their work. But though it may be only a work of the mind, we think of it as real nonetheless. In that universe one can find the never-ending river of real numbers with the integers scattered uniformly along it, the perfect plane of Euclid, many oceans of functions, and the mountains of infinities that build forever on themselves.

But now consider that universe in the light of the Law of Mathematical Unapproachability. While I may visit the mathematical universe and tinker with a few of the pebbles I find there, most of its substance will be invisible to me. (Invisible, not undetectable. I know the non-computable numbers exist. I simply can't "see" them.) The objects mathematicians love to explore are, in fact, for the most part, not within their reach.[8]

The predominance of the transcendental numbers (and its later extension to non-computable numbers) was merely the first proved instance of the Law of Mathematical Unapproachability. Among its many other known occurrences are the following:

- Most continuous functions are hopelessly non-differentiable. This is calculus-speak for saying that most functions have graphs that are indescribably crinkly. In other words, our calculus applies in only a tiny corner of the universe of functions. Yet we study calculus because we can say something about that tiny corner, whereas we have only a few strained examples of what lies outside it.

- Most two-dimensional shapes are fractal, exhibiting infinitely complex behavior viewed at any scale. Traditional plane geometry says little about these objects, and the relatively new field of fractal geometry barely scratches the surface.

- Most mathematical sets cannot be described by any TM, even though sets are, in many ways, the most fundamental objects in mathematics. This limitation puts most of the objects that make up the foundation of mathematics beyond the reach of TMs—presumably, even those within our own skulls.

These instances of the Law of Mathematical Unapproachability are interesting, but they are of limited use in determining limits to human thought. After all, for the most part, they simply say that we can prove the existence of objects with complexity too great for human description. But note those important words:

"we can prove . . ." One could argue that, to a certain degree, we *do* understand non-computable numbers. We can prove they exist. We just can't write one down. The above items give us tasks we cannot perform but not questions we cannot answer. However, there *are* questions we cannot answer. Their existence was guaranteed in one of the most famous mathematical/philosophical developments of the twentieth century: Gödel's "Incompleteness Theorems."

Gödel's Theorems: Mathematics Discovers Its Limitations

In 1931, Kurt Gödel published his now-famous theorems on axiom systems. The exact statements of Gödel's theorems are quite technical, but it is possible to lay out the main ideas in simple terms. Recall that an axiom is an assumption—something we agree to accept as true without proof. An axiom system is a set of such assumptions from which we hope to derive a set of useful theorems. An axiom system is said to be inconsistent if it is possible to prove contradictory statements from its axioms—clearly something we want to avoid. If the axioms have no such built-in contradictions, then we say the axiom system is consistent.

Now axiom systems are somewhat stuffy and hard to think about, so let's switch over to thinking about computing machines. There's actually an easy correspondence between an axiom system and a computing machine. Imagine loading your set of axioms into a machine's memory, programming it to use correct logical inference, and then setting it to the task of outputting a list of all possible theorems that can be proved from those axioms.[9] If the axioms are consistent, the machine will never output two contradictory statements, so we can consider the machine to also be consistent.

In the early twentieth century, there were high hopes that all of mathematics (and perhaps all of the sciences as well) would eventually be axiomatized. If that happened, and if this hypothetical computing machine were constructed to work with those axioms, there would be no more need for mathematicians. If you had a mathematical question, you'd simply ask UMTG (the Universal Math Theorem Generator). But then why stop there? If the sciences are also axiomatized (and human behavior and aesthetics along with them) we could build UEO (the Universal Everything

Oracle) that could predict all events, write the elusive perfect novel, and in short, leave nothing for us to do.[10]

Fortunately for all of us, this will never happen, for Gödel's first theorem says that no such machine is possible. In fact, no consistent machine can generate all theorems in just the limited area of arithmetic of the natural numbers. No matter what axioms you build into your machine, either it will be inconsistent or there will be correct statements about arithmetic that the machine can never derive.

The idea behind Gödel's proof is surprisingly simple. Imagine that we have a set of axioms (call it A), and from it we build a machine M(A) that we claim is a UMTG. Gödel can prove us wrong by constructing a true statement in arithmetic that our machine will never prove. He does this by asking to see how our machine works (that is, he asks to see our axioms A); and from the answer, he produces an arithmetic statement S that (in a complicated but very exact way) encodes the sentence "The machine M(A) will never prove this statement to be true." Now, think about that sentence for a minute:

- If our machine proves Gödel's arithmetic statement S, the sentence becomes false, which because of the encoding, makes statement S false. In this case, our machine is inconsistent since it has proved a false statement to be true.

- On the other hand, our machine certainly can't prove S to be false, for the minute it does, the sentence becomes true. Again, because of the encoding, this makes statement S true. Our machine is again inconsistent, having proved a true statement to be false.

- Thus, if our machine is consistent, the only possibility is that it will say nothing about statement S. But that makes Gödel's sentence true, and thus Gödel's arithmetized version S is a correct arithmetic statement. So we then have an example of a correct arithmetic fact that our machine cannot prove. This proves Göd-el's first theorem.

Gödel's second theorem is similar, but with a slight twist. It says that one thing a consistent axiom system (or computing machine, if you prefer) can never prove is its own consistency. That's a nice bit of logical irony—no consistent computing machine I de-

sign can ever prove the statement "This machine is consistent." In fact, there are only two possibilities for the status of a TM equipped to do arithmetic: Either it will be inconsistent (and thus useless), or it will be unable to demonstrate that it is consistent. Consider what that means for today's mathematics. We do, in fact, have a set of axioms we use as the basis of arithmetic. Gödel's second theorem says that either (1) those axioms are inconsistent—flawed by self-contradiction, or (2) we'll never know that they are not. Those are the only two possibilities.

And, of course, the only way we could ever find out which possibility actually happens is for things to go just as we don't want! It's altogether within the realm of possibility that we could wake up tomorrow to the news that someone somewhere has discovered a contradiction in arithmetic, proving Possibility 1 true. This would be disastrous. Pretty much all of mathematics rests on the properties of the real numbers; so if arithmetic goes, the whole castle comes down. And if mathematics crumbles, what science would remain standing? The best we can hope for is that Possibility 2 is correct. We can't prove it, so we have to hope for it. It's a matter of faith[11] and simple pragmatism. Mathematicians act on the assumption that our axioms must be consistent, though thanks to Gödel, we know we can never be certain.

To summarize, then, Gödel's theorems tell us two things about the limitations of mathematics:

- We can never discover all correct mathematical facts.

- We can never be certain that the mathematics we are doing is free of contradictions.

Mathematicians have grown more or less accustomed to these limitations. Most of us ignore the second one, since it's a matter of faith, and there's nothing much we can do about it. The first one intrigues us because mathematicians love unsolved problems. We're happy that there is a never-ending supply of them.

There are many examples of conjectures in current mathematics that most mathematicians believe are almost certainly true but which seem to elude proof. Perhaps some of them are in fact unprovable (at least with our current axioms) and, hence, are instances of Gödel's first theorem. That wouldn't bother us too
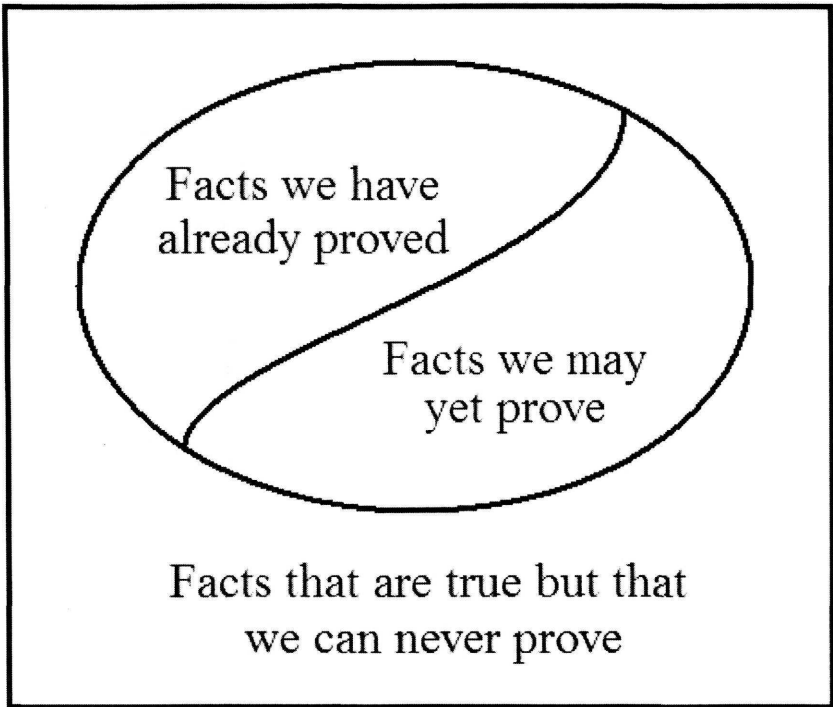
The Universe of Mathematical Proof

much. But we are prone to thinking, consciously or not, that the unprovable facts are strange exceptions and that the ones we *can* prove are the rule. After all, we only know of a few genuinely unprovable statements, so surely (we think) there must be only a few of them. In short, our natural tendency as human mathematicians is to assume that nearly all the mathematics problems we encounter have solutions within the reach of human reason. But remember the transcendental numbers! We know of only a few, but they are in fact the rule. The algebraic numbers are the exceptions! What if the Law of Mathematical Unapproachability applies to mathematical truths?

Our Place in the Universe of Truth

Consider the set of all correct mathematics theorems—the Universe of Mathematical Truth. Once we decide on some axioms

to use, that universe divides naturally into three parts as illus-
trated in the diagram: (1) facts for which we already have proofs,
(2) facts that have proofs we haven't found yet, and facts (we know
they exist thanks to Gödel's first theorem) that have no proofs
from our axioms but that are nonetheless true. Might not the Law
of Mathematical Unapproachability suggest that most things in
that universe fall into the third category? Might it not be that the
"unprovable" part of the universe is in fact nearly everything, with
the other two regions making up only an insignificantly thin slice?
I don't know if that's correct. (Even if it *is* correct, that fact itself is
probably one of those unprovable statements!) I don't even know
the best way to measure the meaning of "most" in this setting. But
I have a gut-level suspicion that something like this is what we're
up against. In fact, I suspect that this picture holds no matter what
axioms we use. Gödel tells us that no choice of axioms will elimi-
nate the existence of unprovable truths. I suspect the natural ex-
tension holds: No choice of axioms can eliminate the predom-
inance of unprovable truths.

Now I want to consider how these ideas from mathematics
might apply to knowledge in general. I have always viewed learn-
ing in general, and mathematics in particular, as an adventure—
something akin to exploring a world. The analogy of a universe of
facts is not really an analogy to me. As a confirmed Platonist I be-
lieve in a universe of all truth—a collection of "that which is."

And I find hints of this Platonist view reflected in LDS scrip-
ture: "Truth abideth and hath no end," we read in Doctrine and
Covenants 88:66. More pointedly, "All truth is independent in
that sphere in which God has placed it, to act for itself, as all intel-
ligence also; otherwise there is no existence" (D&C 93:30). In
Mormon theology, truth is eternal and exists independent of our
ability to detect or derive it. It is absolute.

Given that we exist within such a Universe of Truth, how do
we go about finding our way about within it? I have viewed mathe-
matics as a vehicle I can use in exploring (part of) the Universe of
Truth. But in fact, the vehicle I call mathematics is one we all use
in our exploration. TMs and axiom systems are really just fancy
ways of describing the reasoning processes we all use. Perhaps
you don't use the mathematical language that I do, and perhaps

you are not interested in the more esoteric mathematical land-forms that fascinate me in the Universe of Truth, but we all use deductive reasoning as one way to reach truth, so Gödel's theorems caution us all that there are places this particular vehicle can never take us. In fact, I suspect that the Universe of Truth is a wild and rugged land, and our deduction-driven low-clearance vehicle of conscious human thought can take us to only an insignificant part of it.

The idea that there are truths beyond our reach would not surprise anybody. However, most of us are probably prone to thinking of this limitation as one of volume rather than of substance. We can readily see that there is more information out there than our minds can possibly hold. But the mathematics we have outlined suggests an awesome depth to the picture. There is truth—perhaps most truth, perhaps even almost all truth—that is of an essence and nature beyond our ability to consciously comprehend.

What if the Law of Mathematical Unapproachability is indeed valid and is, furthermore, only a shadow of the larger picture of our position in the Universe of Truth? In that case, the knowledge we are able to obtain through our conscious reasoning would be as sparse in the true substance of truth as the computable numbers are sparse in the real numbers. Almost all objects in the Universe of Truth would defy description or approach by our puny intellects. We might think of labeling bits of truth as either "logical" (capable of being deduced by linear reasoning) or "beyond logic." If my suspicion is correct, almost everything in the Universe of Truth fits into the "beyond logic" category, but the few scattered "logical" bits are most of what we can see. Of course, the greatest truths—the most precious gems in that universe—are probably of the "beyond logic" category. (This would give new meaning to the familiar phrase "It's only logical.")

All of these possibilities run counter to the common tendency to believe in the inevitable ultimate triumph of the human intellect (the same tendency that led the mathematicians in Newton's wake to assume that all things would become predictable through calculus), but it runs in perfect harmony with several scriptural themes, such as the well-known dictum:

> My thoughts are not your thoughts, neither are your ways my ways, saith the LORD.
>
> For as the heavens are higher than the earth, so are my ways higher than your ways, and my thoughts than your thoughts. (Isa. 55:8–9)

I see hints of this theme in the Pearl of Great Price account of Moses's vision. Moses is overwhelmed by what he sees of God's creations. Furthermore, God declares that "they cannot be numbered unto man; but they are numbered unto me, for they are mine" (Moses 1:37) and "Here is wisdom and it remaineth in me" (v. 31). Similar wording is found in Joseph Smith's reaction to his own vision of God's kingdoms: "The mysteries of his kingdom . . . surpass all understanding" (D&C 76:114). The book of Ecclesiastes grapples with the issue of our inability to gain understanding, only to come to the less-than-satisfying conclusion that we cannot ever understand:

> All this have I proved by wisdom: I said, I will be wise; but it was far from me.
>
> That which is far off, and exceeding deep, who can find it out? (Eccl. 7:23–24)

Perhaps, though, the haunting feeling of inadequacy we sometimes get—the one I associate with reading Ecclesiastes, being surrounded by desert mountains, or looking into a star-filled night sky—is also ultimately hopeful, because it tells us something about our capacity to sense that which we cannot know. Perhaps that sense of smallness comes from our spirit's sense of how limited our vision truly is.

In fact, there is reason to suspect such a spiritual ability. For certainly God has access to the totality of the Universe of Truth. His ways, higher than ours as the heavens are higher than the earth, allow Him to see what we cannot. Through what means does He do this? Although God's ability to use reason and deduction would exceed our own (again, heaven and earth is no doubt an apropos analogy), Gödel's theorems place limits on what can be obtained through *any* deductive process, whether that deduction is being performed by man, machine, or even God. God must have access to truth through some greater, non-deductive means. I suspect that our reasoning and logic are but a shadow of a

greater spiritual sense for truth—one that we glimpse here through our personal testimonies. Truth, independent in its sphere, is garnered through spiritual means in greater measure than the trickle we obtain through our linear reasoning. Indeed, most truth is inaccessible to deduction and can only be obtained through this greater means. Even now we can "know" far more than we can give reason for. With God, truth simply is. It needs no derivation. So it will be one day for us.

This possibility helps me make sense of this passage regarding the relationship between our efforts to learn in this life and our ability to acquire truth in the hereafter:

> Whatever principle of intelligence we attain unto in this life, it will rise with us in the resurrection.
> And if a person gains more knowledge and intelligence in this life through his diligence and obedience than another, he will have so much the advantage in the world to come. (D&C 130:18–19)

"Intelligence," as used in Mormon scripture, is a word that obviously has profound meaning. But given its sparse explanation there (D&C 93:29, 36), its meaning is difficult to grasp. Perhaps it is in some way a measure of our ability to obtain truth. Or perhaps intelligence *is* this greater means for truth-gathering—a means not bound by Gödel's limitations on deductive reasoning. Perhaps it is the very means by which God knows truth. The "principle of intelligence we attain unto in this life" is what will rise with us; the next life will be a continuation of the search for truth we should be engaged in here.

Of course, in the end, I have no firm, final answers. I can only speculate on the meaning of what I felt that day reading Ecclesiastes. Those of us who work in science or mathematics develop very rigid ideas about what "knowing" something means. I "know" many things from the mathematics that I have studied. A continuous one-to-one function from a compact topological space to a Hausdorff topological space has a continuous inverse. I know this, and I love knowing such things. What humans have achieved through deduction is both beautiful and amazing to me. In a different way, I "know" that what I see here is only a dim shadow of what must really be—"through a glass, darkly" as Paul puts it—but I "know" that some day I will see it all "face to face" (1 Cor. 13:12).

The first kind of knowing I can explain. It is that of a Turing machine; and given enough time and paper, I could transmit this knowing to you. Not so with the latter type. I cannot explain it even to myself. It is unearthly and mysterious. It is the distant land faintly visible to my spirit from here on the shoreline where conscious deduction ends. The gulf between here and there is, I believe, what gives our spirits such pause in those moments of this life when we confront it.

Notes

1. Kent A. Bessey, "To Journey beyond Infinity," *BYU Studies* 43, no. 4 (2004): 23–32, is an interesting discussion of the philosophical implications of infinity.

2. There is actually a subtle mathematical twist here. We would like "equal cardinality" to mean that the two sets can be put into exact one-to-one correspondence. But $|Y| \geq |X|$ and $|X| \geq |Y|$ mean only that we can put X in one-to-one correspondence with part of Y and Y in one-to-one correspondence with part of X. Proving that these two conditions imply an exact one-to-one correspondence between X and Y was difficult enough to stump Cantor, but successful proofs were eventually produced by several mathematicians independently.

3. Named for Alan Turing (1913–54), one of the founders of the theory of computation, who first championed the TM concept as a model for algorithmic processes.

4. The three-millionth digit of π after the decimal point is a 3. In fact, digits 3,000,000 through 3,000,009 in π are 3697067915.

5. Actually, one really wouldn't want to use *this* particular formula, since it converges much too slowly to π. There are other similar (but more complicated) formulas that give much faster results.

6. Belief in this universe of mathematical objects is central to the Platonist philosophy in mathematics. A Platonist mathematician believes that the mathematical objects he or she studies—the integers, the real numbers, functions, shapes, and so on—actually exist, and that his or her work as a mathematician consists of discovering the properties of these objects. In contrast, the Formalist philosophy holds that mathematics is a human invention and that mathematical terms are simply abstract constructs having no real existence. Most mathematicians have a bit of both schools in them and are quite comfortable switching back and forth between the two outlooks as occasion requires, in much the same way that physicists have become comfortable with thinking of light as particle and/or wave.

7. Ian Stewart's *Flatterland* (Cambridge, Mass.: Perseus Publishing, 2001), a modern follow-up to Edwin Abbot's classic *Flatland,* gives the best description I have seen of the mathematical universe or, in his terminology, "Mathiverse." (See pp. 28–30.)

8. Though he lived two centuries before the central ideas discussed in this paper began to emerge, Sir Isaac Newton expressed something very like this sentiment in one of his most famous quotations. Shortly before his death, he wrote in his memoirs: "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." David Brewster, *Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton* (Edinburgh, Scotland: n.pub., 1855), vol. 2, chap. 27.

9. Though it sounds like every geometry student's dream, it isn't difficult to describe—in theory anyway—how such a machine would work. It would begin by outputting all conclusions reached by "one-step proofs"—conclusions reached by quoting one axiom. Then by beginning with these statements and following with each of the system's axioms in turn, it can list all of the conclusions reached by "two-step proofs." These would then allow easy computation of the conclusions of "three-step proofs," and so on. Continuing in this way, any theorem that can be proved from the axioms would eventually be output.

10. This section follows ideas from Rudy Rucker's excellent book, *Infinity and the Mind: The Science and Philosophy of the Infinite* (Princeton, N.J.: Princeton University Press, 1995), chap. 4.

11. Even the most ardently atheistic or agnostic mathematicians and scientists, then, must be practitioners of the principle of faith. Without faith in the (unprovable) consistency of our mathematics, there would not be much point to pursuing mathematical or scientific questions.